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An Elementary Proof of the Oscillation Lemma for Weak Markov Systems

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Let M be a subset of the real line, and

$$\Delta_k(M) = \{(t_1, t_2, ..., t_k) \in M^k \mid t_1 < t_2 < \cdots < t_k\} \qquad \text{for} \quad k \in \mathbb{N}.$$

Let $F(M)$ be the set of real-valued functions defined on M, $f_0, f_1, ..., f_n \in F(M)$ fixed and linearly independent, and $U_i = \text{lin}\{f_0, f_1, ..., f_i\}$, the linear span of $\{f_0, f_1, ..., f_i\}$, for $i = 0, 1, ..., n$. The sequence $f_0, f_1, ..., f_n$ is called a weak Markov system iff for each $k \in \{0, 1, ..., n\}$, $\det(f_i(t_j))_{i,j=0}^k$ has weakly constant sign for all $(t_0, t_1, ..., t_k) \in \Delta_{k+1}(M)$, or, equivalently, iff for each $k \in \{0, 1, ..., n\}$, no $f \in U_k$ has a strong alternation of length $k+2$; i.e., there is no $(t_0, t_1, ..., t_{k+1}) \in \Delta_{k+2}(M)$ with $f(t_i) \cdot f(t_{i+1}) < 0$ for $i = 0, 1, ..., k$. If $f_0 \equiv 1$, the system is called normalized.

Generalizing a result of D. Zwick [3], in [1] we proved the following result:

LEMMA. *If $f_0, f_1, ..., f_n$ form a normalized weak Markov system, no $f \in U_n$ has a strong oscillation of length $n+2$; i.e., there is no $(t_0, t_1, ..., t_{n+1}) \in \Delta_{n+2}(M)$ with $[f(t_k) - f(t_{k-1})] \cdot [f(t_{k+1}) - f(t_k)] < 0$ for $k = 1, ..., n$.*

The proof was based on the Gauss kernel approximation of weak Markov systems by Markov systems and the oscillation lemma for normalized Markov systems (Lemma 8.7a in [2]). It was, however, pointed out independently by several authors[1] that the Gauss kernel concept does not seem to be needed anywhere else in the basic theory of weak Čebyšev or Markov systems.

We shall subsequently present an elementary proof of the above lemma.

We proceed by induction over n. For $n \in \{0, 1\}$ the statement is obvious. Let us assume it holds for $n-1$ and suppose it fails for n. So there exist $f \in U_n \setminus U_{n-1}$ and $(t_0, ..., t_{n+1}) \in \Delta_{n+2}(M)$ with $f(t_0) > f(t_1) < f(t_2) > \cdots$.

[1] Oral communications by M. Sommer, D. Zwick and others.

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We distinguish several cases and subcases.

Case 1. $\dim U_{n-1}|_{\{t_0, ..., t_{n+1}\}} = n$.

Subcase 1a. $\dim U_{n-1}|_{\{t_1, ..., t_n\}} = n-2$. This implies

$$\dim U_{n-1}|_{\{t_0, ..., t_n\}} = n-1.$$

For $h \in F(M)$, let us denote by $\hat{h}$ the restriction of h to $\{t_0, ..., t_n\}$. As $\hat{f}_0, ..., \hat{f}_{n-1}$ are linearly dependent, there is a minimal j with $\hat{f}_j \in \text{lin}\{\hat{f}_0, ..., \hat{f}_{j-1}\}$, say $\hat{f}_j = \sum_{i=0}^{j-1} \alpha_i \hat{f}_i$, $\alpha_0, ..., \alpha_{j-1} \in \mathbb{R}$. We claim that $\hat{f}_0, ..., \hat{f}_{j-1}, \hat{f}_{j+1}, ..., \hat{f}_n$ is a (normalized) weak Markov system. Indeed, suppose some $\hat{g} \in \text{lin}\{\hat{f}_0, ..., \hat{f}_{j-1}, \hat{f}_{j+1}, ..., \hat{f}_k\}$ has a strong alternation of length $k+1$, say, in $t_{i_0}, ..., t_{i_k}$ with $0 \leqslant i_0 < \cdots < i_k \leqslant n$. We have $\dim U_j|_{\{t_0, ..., t_{n+1}\}} \geqslant j+1$, and so for $h := f_j - \sum_{i=0}^{j-1} \alpha_i f_i$ we get: $h(t_0) = \cdots = h(t_n) = 0 \neq h(t_{n+1})$. But then $g + \gamma h \in U_k$ has a strong alternation of length $k+2$ in $t_{i_0}, ..., t_{i_k}$, t_{n+1} for suitable $\gamma \in \mathbb{R}$, a contradiction.

Applying the induction hypothesis to $\hat{f}_0, ..., \hat{f}_{j-1}, \hat{f}_{j+1}, ..., \hat{f}_n$, we see that $\hat{f}$ cannot have a strong oscillation of length $n+1$ in $t_0, ..., t_n$, and we arrive at a contradiction.

Subcase 1b. $\dim U_{n-1}|_{\{t_1, ..., t_n\}} = n-1$. If we have

$$\dim U_{n-1}|_{\{t_0, ..., t_n\}} = n-1 \qquad \text{or} \qquad \dim U_{n-1}|_{\{t_1, ..., t_{n+1}\}} = n-1,$$

the argument is the same or analogous to Subcase 1a. So let us assume

$$\dim U_{n-1}|_{\{t_0, ..., t_n\}} = \dim U_{n-1}|_{\{t_1, ..., t_{n+1}\}} = n.$$

Now let $r \in \{1, ..., n\}$ be chosen such that

$$\dim U_{n-1}|_{\{t_1, ..., t_{r-1}, t_{r+1}, ..., t_n\}} = n-1.$$

So we have

$$\dim U_{n-1}|_{\{t_0, t_1, ..., t_{r-1}, t_{r+1}, ..., t_n\}} = n,$$

and can define a basis $g_0, ..., g_{r-1}, g_{r+1}, ..., g_n$ of U_{n-1} by

$$g_i(t_j) = \delta_{i,j} \qquad \text{for} \quad i, j \in \{0, 1, ..., r-1, r+1, ..., n\}.$$

Now $g_0(t_r) \neq 0$ would imply that $g_0, ..., g_{r-1}, g_{r-1}, ..., g_n$ are linearly independent on $\{t_1, ..., t_n\}$, contradicting $\dim U_{n-1}|_{\{t_1, ..., t_n\}} = n-1$. So we have $g_0(t_r) = 0$. This implies $g_0(t_{n+1}) \neq 0$, for otherwise g_0 would vanish on $\{t_1, ..., t_{n+1}\}$, contradicting $\dim U_{n-1}|_{\{t_1, ..., t_{n+1}\}} = n$.

For $\varepsilon \in \mathbb{R}$, we define

$$h_\varepsilon := g_0 + \varepsilon \sum_{i=1}^{r-1} (-1)^i g_i + \varepsilon \sum_{i=r+1}^{n} (-1)^{i+1} g_i.$$

For sufficiently small $\varepsilon > 0$, h_ε has a strong alternation of length n in $t_0, t_1, ..., t_{r-1}, t_{r+1}, ..., t_n$, and sign $h_\varepsilon(t_{n+1}) = \text{sign } g_0(t_{n+1}) \neq 0$.

The alternation property yields sign $h_\varepsilon(t_n) = \text{sign } h_\varepsilon(t_{n+1})$ for $r < n$, and sign $h_\varepsilon(t_{n-1}) = \text{sign } h_\varepsilon(t_{n+1})$ for $r = n$. In either case we obtain:

$$\text{sign } g_0(t_{n+1}) = (-1)^{n-1}. \qquad (*)$$

Now let $g \in U_{n-1}$ be such that g interpolates f in $t_0, ..., t_{r-1}, t_{r+1}, ..., t_n$. If we had $f(t_r) = g(t_r)$, g would have a strong oscillation of length $n+1$, contradicting the induction hypothesis. So we have $(f-g)(t_r) \neq 0$, and for sufficiently small $\alpha > 0$,

$$d_\alpha := f - g + \alpha \cdot \text{sign}((f-g)(t_r)) \sum_{\substack{i=1 \\ i \neq r}}^{n} (-1)^{r-i} g_i$$

has a strong alternation of length n in $t_1, ..., t_n$. From $(*)$ we conclude that for a suitable $\gamma \in \mathbb{R}$, $d_\alpha + \gamma g_0$ has a strong alternation of length $n+2$ in $t_0, ..., t_{n+1}$, a contradiction.

Subcase 1c. $\dim U_{n-1}|_{\{t_1, ..., t_n\}} = n$. For sufficiently small $\varepsilon > 0$, $g_\varepsilon \in U_{n-1}$ defined by

$$g_\varepsilon(t_i) = \begin{cases} f(t_i) + \varepsilon & \text{for } i \text{ odd} \\ f(t_i) - \varepsilon & \text{for } i \text{ even} \end{cases} \quad i = 1, ..., n,$$

has a strong oscillation of length n in $t_1, ..., t_n$, and the induction hypothesis implies $g_\varepsilon(t_0) \leqslant g_\varepsilon(t_1) = f(t_1) + \varepsilon < f(t_0)$ and $g_\varepsilon(t_{n+1}) \leqslant g_\varepsilon(t_n) = f(t_n) + \varepsilon < f(t_{n+1})$ if n is odd, $g_\varepsilon(t_{n+1}) \geqslant g_\varepsilon(t_n) - \varepsilon > f(t_{n+1})$ if n is even. So $f - g_\varepsilon$ has a strong alternation of length $n+2$ in $t_0, ..., t_{n+2}$, a contradiction.

Case 2. $\dim U_{n-1}|_{\{t_0, ..., t_{n+1}\}} \leqslant n-1$. Then in a way completely analogous to the proof of Lemma 4.1, part (b) $\Rightarrow$ (a), Case 2 in [2], it can be shown that there exists $(u_0, ..., u_{n+1}) \in \Delta_{n+2}(M)$ with $\dim U_{n-1}|_{\{u_0, ..., u_{n+1}\}} > \dim U_{n-1}|_{\{t_0, ..., t_{n+1}\}}$, forming a strong oscillation of f of length $n+2$, and after finitely many repetitions of this argument one arrives at Case 1.

References

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